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A Few Properties of a Certain Class of Degenerate Space–Times

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Abstract

The properties are studied of a class of space–times determined by assuming the shape of the metric form ds^2 including disposable coordinate functions. It has been found that this class includes degenerate space–times with geodetic, null, shear-free congruences with nonvanishing expansion. The theorem has been proved that this class of solutions of the Einstein equations can easily be expanded to solutions of Einstein–Maxwell equations with a fairly general electromagnetic field. For a selected subclass relations are given between the functions determining the metric form, and two new explicit solutions with arbitrary functions of the Einstein–Maxwell equations with a cosmological constant are found.

1. *Method and Formalism*

When presenting some of the results and in the calculations the tetrad formalism was used described by Debney et al. (1969). For convenience this formalism will be presented here in a very abbreviated form.

In the whole space–time the tetrad field of independent $e_a{}^\mu (a, \mu = 1, 2, 3, 4)$ vectors is introduced, and it is assumed that the Greek letters are suffixes referring to the tensor formalism and the Latin letters to the tetrad formalism. Summation convention holds for both kinds of suffixes. The $e^a{}_\mu$ vectors are given by the relations

$$e_a{}^\mu e^b{}_\mu = \delta_a{}^b, \qquad e_a{}^\mu e^a{}_\nu = \delta_\nu{}^\mu \tag{1.1}$$

The scalar $T_{a\cdots}{}^{b\cdots}$ is referred to as tetrad component of the $T_{\mu\cdots}^{\nu\cdots}$ tensor. The following relations occur:

$$T_{a\cdots}{}^{b\cdots} \underset{\text{def}}{=} e_a{}^\mu e^b{}_\nu \ldots T_{\mu\cdots}{}^{\nu\cdots}, \qquad T_{\mu\cdots}{}^{\nu\cdots} = e^a{}_\mu e_b{}^\nu \cdots T_{a\cdots}{}^{b\cdots} \tag{1.2}$$

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The notation e^a is used to denote the differential form of the first order:

$$e^a \underset{\text{def}}{=} e^a{}_\mu \, dx^\mu \tag{1.3}$$

and the metric form can be presented as follows:

$$ds^2 = e^a e_a = g_{\mu\nu} \, dx^\mu \, dx^\nu \tag{1.4}$$

The Ricci rotation coefficients $\Gamma^a{}_{bc}$, which play a role in the tetrad covariant derivative expressions analogous to that of the connection coefficients in the tensor covariant derivative ones are as follows:

$$\Gamma^a{}_{bc} = -e^a{}_{\mu;\nu} e_b{}^\mu e_c{}^\nu \tag{1.5}$$

The relations between the $e^a{}_\mu$ vectors, $\Gamma^a{}_{bc}$ coefficients, and tetrad components of the Riemann tensor $R^a{}_{bcd}$ are given by the Cartan formulas

$$de^a = e^b \wedge \Gamma^a{}_b = \Gamma^a{}_{bc} e^b \wedge e^c \tag{1.6}$$

$$d\Gamma^a{}_b + \Gamma^a{}_m \wedge \Gamma^m{}_b = \tfrac{1}{2} R^a{}_{bcd} e^c \wedge e^d \tag{1.7}$$

where

$$\Gamma^a{}_b \underset{\text{def}}{=} \Gamma^a{}_{bc} e^c \tag{1.8}$$

and the notation d and $\wedge$ denotes the exterior derivative operator and exterior multiplication, respectively.

The tetrad system of $e_a{}^\mu$ vectors has been assumed so that the e_1 and e_2 forms are mutually complex and conjugate, and the e_3 and e_4 forms are real, viz.,

$$e_1 = \bar{e}_2, \qquad e_3 = \bar{e}_3, \qquad e_4 = \bar{e}_4 \tag{1.9}$$

and that

$$g_{ab} = e_a{}^\mu e_{b\mu} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = g^{ab} \tag{1.10}$$

thus all the $e_a{}^\mu$ vectors are null vectors.

The tetrad system determined by (1.9) and (1.10) is convenient for the calculations since the raising or lowering of the tetrad suffixes reduces to their variation according to the rule $1, 2, 3, 4 \to 2, 1, 4, 3$, and the operation of obtaining conjugate quantities consists in their variation according to the rule $1, 2, 3, 4 \to 2, 1, 3, 4$, e.g., $k_1{}^4 = k^2{}_3 = \bar{k}^1{}_3 = k_{23}$.

Condition (1.10) is equivalent to the metric form having the shape

$$ds^2 = 2e^1 e^2 + 2e^3 e^4 \tag{1.11}$$

In the selected tetrad system given by (1.9) and (1.10) we have

$$\Gamma_{abc} = -\Gamma_{bac} \tag{1.12}$$

where

$$\Gamma_{abc} \underset{\text{def}}{=} g_{am}\Gamma^{m}{}_{bc} \tag{1.13}$$

and there exist three independent differential forms of first order, viz., Γ_{42}, $\Gamma_{12} + \Gamma_{34}$ and Γ_{31}. The Ricci tensor has seven independent components: four real $R_{12}, R_{34}, R_{33}, R_{44}$ and three complex R_{22}, R_{23}, R_{24}. The following convention has been assumed:

$$R_{ab} = R^{c}{}_{abc} \tag{1.14}$$

The curvature scalar $R = R^{a}{}_{a}$ is expressed as follows:

$$R = 2R_{12} + 2R_{34} \tag{1.15}$$

The condition $\Gamma_{424} = 0$ is equivalent to the vector $e_4{}^{\mu}(= e^{3\mu})$ being geodetic. The condition $\Gamma_{424} = \Gamma_{422} = 0$ is equivalent to the vector $e_4{}^{\mu}$ being geodetic and shear-free. If $\Gamma_{424} = 0$, then

$$Z \underset{\text{dif}}{=} \Theta + i\omega = -\Gamma_{421} \tag{1.16}$$

where Θ is the expansion and ω the rotation of vector $e^{3\mu}$. The quantity Z is known as the complex expansion. If in these relations we change the suffixes in Γ_{abc} according to the rule $4 \to 3$ and $2 \leftrightarrow 1$, then the relations will refer to the vector $e_3{}^{\mu}(= e^{4\mu})$.

Weyl's conformal curvature tensor can be characterized by five complex quantities $C^{(i)}(i = 1, 2, 3, 4, 5)$. A space-time is conformally flat if and only if all $C^{(i)}$ vanish. If a space-time is not conformally flat, then the conditions for $e_4{}^{\mu}$ to be a Debever-Penrose null vector, single, double, triple or quadruple are $C^{(5)} = 0$, $C^{(5)} = C^{(4)} = 0$, $C^{(5)} = C^{(4)} = C^{(3)} = 0$ or $C^{(5)} = C^{(4)} = C^{(3)} = C^{(2)} = 0$, respectively; for vector $e_3{}^{\mu}$ similarly $C^{(1)} = 0$, $C^{(1)} = C^{(2)} = 0$, etc. The quantities $C^{(i)}$ can be expressed in a simple way by means of quantities R_{abcd} (see Debney et al., 1969).

The following lemma has been proved:

> *Lemma 1.* If $C^{(5)} = C^{(4)} = 0$ and $C^{(3)} \neq 0$, then the space-time is of [2, 2] Petrov type if and only if $3C^{(1)}C^{(3)} - 2[C^{(2)}]^2 = 0$.

This lemma also holds when we change the suffixes i in $C^{(i)}$ according to the rule $5, 4, 3, 2, 1 \to 1, 2, 3, 4, 5$.

In the present work units have been selected so that the speed of light and the constant of gravitation are equal to unity: and signature $+ + + -$.

2. *Shape of the Metric Form ds^2 and the Theorem of Expansion for Solutions with Electromagnetic Field*

We studied the class of metric forms $ds^2 = 2e^1e^2 + 2e^3e^4$ given by the equations

$$e^1 = \frac{r + i\beta}{\alpha}\, dY = \overline{e^2} \tag{2.1a}$$

$$e^3 = d\rho + \sigma\, dY + \bar{\sigma}\, d\bar{Y} \tag{2.1b}$$

$$e^4 = dr - i\, \partial_Y \beta\, dY + i\, \partial_{\bar{Y}} \beta\, d\bar{Y} + \gamma\, e^3 \tag{2.1c}$$

where r and ρ are real coordinates, and Y is a complex coordinate, and besides

$$\sigma \underset{\text{def}}{=} 2m\, \partial_Y \ln \alpha - i\epsilon\, \partial_Y \beta = \sigma(Y, \bar{Y}) \tag{2.1d}$$

$$\gamma \underset{\text{def}}{=} \frac{1}{2}\left[-\epsilon + \frac{(m+\bar{m})r + i\beta(\bar{m}-m) + f}{r^2 + \beta^2}\right] = \bar{\gamma} = \gamma(r, \rho, Y, \bar{Y}) \tag{2.1e}$$

where coefficient ϵ and the disposable functions α, β, f, m are limited by the conditions

$$\epsilon = \pm 1 \tag{2.1f}$$

$$\alpha = \alpha(Y, \bar{Y}) = \bar{\alpha} \neq 0 \tag{2.1g}$$

$$\beta = \beta(Y, \bar{Y}) = \bar{\beta} \tag{2.1h}$$

$$f = f(r, \rho, Y, \bar{Y}) = \bar{f} \tag{2.1i}$$

$$m = m(Y) \tag{2.1j}$$

Additionally we introduce a limiting condition connecting the functions α, β, and m:

$$\alpha^2[(\bar{m} - m)\, \partial_Y \partial_{\bar{Y}} \ln \alpha + i\epsilon\, \partial_Y \partial_{\bar{Y}} \beta] + i\beta = 0 \tag{2.1k}$$

Certain limitations ensuing from this condition are discussed further in this section in footnote 3.

Although the γ function is a disposable real function of all coordinates we express it by another disposable real function f of all coordinates as shown in (2.1e), since in many cases it is more convenient to employ the function f than γ. In the class of metrics (2.1) function f is given with an accuracy to the component cr, where $c = \bar{c} = \text{const}$, which results from the disposable character of the m function. If we assume $f = f' + cr$, $m = m'(Y) - c/2$ and $\rho = \rho' + c \ln \alpha$, then after dropping the primes we obtain again the expressions (2.1).

From (2.1d) and (2.1k) it follows that

$$\partial_{\bar{Y}} \sigma - \partial_Y \bar{\sigma} = 2i\beta/\alpha^2 \tag{2.2}$$

We have the following lemma:

> *Lemma 2.* If $\partial_Y \partial_{\bar{Y}} \ln \alpha \neq 0$, then the conditions $\beta = 0$ and $\sigma = 0$ are equivalent in the sense that the shape of the metric form ds^2 is preserved.[2]

[2] Under the assumption, if $de^3 = 0$ then there exists ρ such that $e^3 = d\rho$, the conditions $\beta = 0$ and $\sigma = 0$ are equivalent in class (2.1) (in the above given sense) and the assumption $\partial_Y \partial_{\bar{Y}} \ln \alpha \neq 0$ is unnecessary.

Proof. If $\sigma = 0$, it follows immediately from (2.2) that $\beta = 0$ (without the condition $\partial_Y \partial_{\bar{Y}} \ln \alpha \neq 0$). If on the other hand $\beta = 0$, then because $\partial_Y \partial_{\bar{Y}} \ln \alpha \neq 0$ we have from (2.1j) and (2.1k) that $m = \bar{m} = \text{const}$, and hence considering (2.1b) and (2.1d) we get $e^3 = d\rho'$, where $\rho' = \rho + 2m \ln \alpha$.

Let us notice that if $\beta = 0$ and $\partial_Y \partial_{\bar{Y}} \ln \alpha \neq 0$, then σ disappears only in the sense that $e^3 = d\rho$ and despite (2.1d) we may have $m \neq 0$. Hence, for $\beta = 0$ the component $(m + \bar{m})r = \text{const} \cdot r$ in the numerator of the fraction in (2.1e) may be preserved. Of course we can also have $m = 0$, but in this case the component const $\cdot\, r$ in the numerator of the fraction in (2.1e) (related in certain cases with the mass parameter as we shall see in the subsequent section of the present paper) need not disappear in view of the fact that the function f is determined with an accuracy to the component const $\cdot\, r$.

Introducing the operator $\partial_a \underset{\text{dif}}{=} e_a{}^\mu \partial_\mu$ we obtain for class (2.1) that

$$\partial_1 = \alpha Z(\partial_Y - \sigma \partial_\rho + i\, \partial_Y \beta \cdot \partial_r) = \overline{\partial_2} \tag{2.3a}$$

$$\partial_3 = \partial_\rho - \gamma \partial_r \tag{2.3b}$$

$$\partial_4 = \partial_r \tag{2.3c}$$

$$de^1 = \bar{Z} \partial_{\bar{Y}} \alpha \cdot e^1 \wedge e^2 + Z\gamma e^1 \wedge e^3 - Z e^1 \wedge e^4 = \overline{de^2} \tag{2.4a}$$

$$de^3 = -2iZ\bar{Z}\beta e^1 \wedge e^2 \tag{2.4b}$$

$$de^4 = 2iZ\bar{Z}(\alpha^2 \partial_Y \partial_{\bar{Y}} \beta - \beta\gamma) e^1 \wedge e^2 + \partial_a \gamma \cdot e^a \wedge e^3 \tag{2.4c}$$

$$\Gamma_{121} + \Gamma_{341} = Z\, \partial_Y \alpha \tag{2.5a}$$

$$\Gamma_{122} + \Gamma_{342} = -\bar{Z}\, \partial_{\bar{Y}} \alpha \tag{2.5b}$$

$$\Gamma_{123} + \Gamma_{343} = iZ\bar{Z}(2\beta\gamma - \alpha^2 \partial_Y \partial_{\bar{Y}} \beta) + \partial_r \gamma \tag{2.5c}$$

$$\Gamma_{124} + \Gamma_{344} = 0 \tag{2.5d}$$

$$\Gamma_{311} = \Gamma_{314} = 0 \tag{2.6a}$$

$$\Gamma_{312} = Z\gamma + iZ\bar{Z}\alpha^2 \partial_Y \partial_{\bar{Y}} \beta \tag{2.6b}$$

$$\Gamma_{313} = \partial_1 \gamma \tag{2.6c}$$

$$\Gamma_{421} = -(r + i\beta)^{-1} = -Z \neq 0 \tag{2.7a}$$

$$\Gamma_{422} = \Gamma_{423} = \Gamma_{424} = 0 \tag{2.7b}$$

$$R_{12} = -2Z\bar{Z}\alpha_2 \partial_Y \partial_{\bar{Y}} \ln \alpha + {} + (Z\bar{Z})^2 [-2\alpha^2 \beta \partial_Y \partial_{\bar{Y}} \beta + \epsilon(r^2 - \beta^2) + i\beta(\bar{m} - m) + f - r\, \partial_r f] \tag{2.8a}$$

$$R_{34} = -\tfrac{1}{2} Z\bar{Z}\, \partial_r \partial_r f + (Z\bar{Z})^2 [2\alpha^2 \beta\, \partial_Y \partial_{\bar{Y}} \beta + 2\epsilon\beta^2 - i\beta(\bar{m} - m) - f + r\, \partial_r f] \tag{2.8b}$$

$$
\begin{aligned}
R_{23} = \alpha Z\bar{Z}^2 \{ & 2i\alpha\, \partial_{\bar{Y}}\alpha \cdot \partial_Y \partial_{\bar{Y}}\beta + i\alpha^2 \partial_Y \partial_{\bar{Y}} \partial_{\bar{Y}}\beta + \tfrac{1}{2}\partial_{\bar{Y}}\bar{m} \\
& -\tfrac{1}{2}\partial_r \partial_{\bar{Y}} f + \tfrac{1}{2} i\, \partial_{\bar{Y}}\beta \cdot \partial_r \partial_r f + \tfrac{1}{2}\bar{\sigma}\partial_\rho \partial_r f + i\epsilon\, \partial_{\bar{Y}}\beta \\
& -2\bar{Z}\alpha^2 \partial_{\bar{Y}}\beta \cdot \partial_Y \partial_{\bar{Y}}\beta + Z[i(\bar{m} - m)\partial_{\bar{Y}}\beta + \partial_{\bar{Y}} f] - Z\bar{\sigma}\,\partial_\rho f \\
& -2\epsilon\bar{Z}\beta\,\partial_{\bar{Y}}\beta + 2iZ\bar{Z}[i\beta(\bar{m} - m) + f - r\,\partial_r f]\ \partial_{\bar{Y}}\beta \}
\end{aligned} \tag{2.8c}
$$

$$
\begin{aligned}
R_{33} = (\alpha Z\bar{Z})^2 \Bigg\{ & -\left[\frac{r}{\alpha^2} + \frac{1}{2}\,(\partial_Y \bar{\sigma} + \partial_{\bar{Y}}\sigma)\right] \partial_\rho f + \sigma\bar{\sigma}\,\partial_\rho \partial_\rho f \\
& + \sigma(2Z\bar{Z}\beta\,\partial_{\bar{Y}}\beta \cdot \partial_\rho f - \partial_\rho \partial_{\bar{Y}} f) + \bar{\sigma}(2Z\bar{Z}\beta\,\partial_Y\beta \cdot \partial_\rho f - \partial_\rho \partial_Y f) \\
& + 2(\epsilon\,\partial_Y\beta \cdot \partial_{\bar{Y}}\beta + im\,\partial_{\bar{Y}}\beta \cdot \partial_Y \ln\alpha - i\bar{m}\,\partial_Y\beta \cdot \partial_{\bar{Y}}\ln\alpha)(\partial_\rho \partial_r f - 2rZ\bar{Z}\,\partial_\rho f) \\
& - 2\alpha^2(\partial_Y \partial_{\bar{Y}}\beta)^2 - 2\epsilon\beta\,\partial_Y \partial_{\bar{Y}}\beta + i(\bar{m} - m)\,\partial_Y \partial_{\bar{Y}}\beta + \partial_Y \partial_{\bar{Y}} f \\
& + i\,\partial_Y\beta \cdot \partial_r \partial_{\bar{Y}} f - i\partial_{\bar{Y}}\beta \cdot \partial_r \partial_Y f + \partial_Y\beta \cdot \partial_{\bar{Y}}\beta \cdot \partial_r \partial_r f \\
& - 2iZ\,\partial_Y\beta \cdot \partial_{\bar{Y}} f + 2i\bar{Z}\,\partial_{\bar{Y}}\beta \cdot \partial_Y f + 4Z\bar{Z}(f - r\,\partial_r f)\partial_Y\beta \cdot \partial_{\bar{Y}}\beta \Bigg\}
\end{aligned} \tag{2.8d}
$$

$$
R_{22} = R_{24} = R_{44} = 0 \tag{2.8e}
$$

$$
C^{(5)} = C^{(4)} = 0 \tag{2.9a}
$$

$$
\begin{aligned}
C^{(3)} = {} & 2Z[iZ\bar{Z}(2\beta\gamma - \alpha^2 \partial_Y \partial_{\bar{Y}}\beta) + \partial_r\gamma] \\
& - \tfrac{2}{3}Z\bar{Z}(\alpha^2 \partial_Y \partial_{\bar{Y}} \ln\alpha + \gamma + 2r\,\partial_r\gamma) - \tfrac{1}{3}\partial_r \partial_r\gamma
\end{aligned} \tag{2.9b}
$$

$$
C^{(2)} = -\partial_1 [iZ\bar{Z}(2\beta\gamma - \alpha^2 \partial_Y \partial_{\bar{Y}}\beta) + \partial_r\gamma] + 2Z\,\partial_1\gamma \tag{2.9c}
$$

$$
C^{(1)} = -2\,\partial_1 \partial_1\gamma - 2Z\,\partial_Y\alpha \cdot \partial_1\gamma \tag{2.9d}
$$

As is seen from (2.9a) for space-times of (2.1) class the vector $e^{3r \to \mu}$ is at least a double Debever-Penrose vector (provided the space-time is not conformally flat). Hence in class (2.1) there are no algebraically general solutions (of [1, 1, 1, 1] Petrov type). It follows from (2.7b) that the $e^{3\mu}$ vector is geodesic and shear-free, and from (2.7) that the latter vector reveals an expansion Θ different from zero and its rotation ω disappears if and only if $\beta = 0$.[3]

[3] It appears that in class (2.1a)–(2.1j) without condition (2.1k) vector $e^{3\mu}$ is also geodetic and shear-free ($\Gamma_{424} = \Gamma_{422} = 0$) with a nondisappearing expansion Θ. The $e^{3\mu}$ vector is then also the Debever–Penrose vector ($C^{(5)} = 0$), but need not be degenerated since $C^{(4)}$ need not be equal to zero. From this point of view the solutions of the [1, 1, 1, 1] Petrov type are not excluded in class (2.1a)–(2.1j). On the other hand we do not know if in the latter class at least one solution may exist of any Einstein equations if condition (2.1k) is not fulfilled. We do not know, therefore, if condition (2.1k) is independent of (2.1a)–(2.1j) or whether it is a conclusion drawn from the latter equations under the assumption that solutions exist of arbitrary Einstein equations in class (2.1a)–(2.1j). It has been found that if we assume ds^2 in the form (2.1a)–(2.1j), then we obtain that $R_{24} = C^{(4)}$ and $R_{22} = 0$ and that the condition (2.1k) is equivalent to $C^{(4)} = R_{24} = R_{44} = 0$. Hence if we seek for example solutions of equations $R_{\mu\nu} = \lambda g_{\mu\nu}$ in class (2.1a)–(2.1j), then the relation (2.1k) will be a conclusion from these assumptions, and we shall obtain a degenerate solution only.

The electromagnetic field of interest to us is determined by means of the vector potential V_μ, given by the differential form $V = V_\mu dx^\mu$, in the following way:

$$V = -\tfrac{1}{2}(eZ + \bar{e}\bar{Z})e^3 - \tfrac{1}{2}dY \cdot \int \bar{e}\alpha^{-2}\, d\bar{Y} - \tfrac{1}{2}d\bar{Y} \cdot \int e\alpha^{-2}\, dY \tag{2.10}$$

where

$$e = e(Y) \tag{2.11}$$

is an arbitrary analytical function of the variable Y, only.

Vector V_μ as potential is of course given with an accuracy up to the gradient of the arbitrary function, and hence the form V is determined with an accuracy to the derivative of that function.

The tetrad components F_{ab} of the electromagnetic field tensor $F_{\mu\nu}$ corresponding to the potential V_μ are found from the equation

$$dV = -\tfrac{1}{2}F_{ab}e^a \wedge e^b \tag{2.12}$$

and the four independent among them are

$$F_{12} = \tfrac{1}{2}eZ^2 - \tfrac{1}{2}\bar{e}\bar{Z}^2 \tag{2.13a}$$

$$F_{34} = \tfrac{1}{2}eZ^2 + \tfrac{1}{2}\bar{e}\bar{Z}^2 \tag{2.13b}$$

$$F_{23} = \alpha\bar{Z}^2(\tfrac{1}{2}\,\partial_{\bar{Y}}\bar{e} + i\bar{e}\bar{Z}\,\partial_{\bar{Y}}\beta) = \tfrac{1}{2}\,\partial_2(\bar{e}\bar{Z}) \tag{2.13c}$$

$$F_{24} = 0 \tag{2.13d}$$

The electromagnetic field given by relations (2.13) fulfills the full set of Maxwell equations without the currents and charges:

$$d(F_{ab}e^a \wedge e^b) = 0 \tag{2.14a}$$

$$F^{ab}{}_{;b} = 0 \tag{2.14b}$$

in *every* space-time characterized by the metric of class (2.1). Of course (2.14a) follows immediately from (2.12). The direction $e^{3\mu}$ is the principal null direction of the electromagnetic field (2.13), since

$$e^3{}_{[\mu}F_{\nu]\tau}e^{3\tau} = 0 \tag{2.15}$$

Let us introduce the notation

$$f_1 \underset{\text{def}}{=} f - e\bar{e} \tag{2.16a}$$

If we now assume that $ds_1{}^2$ is the metric that is formed from the metric (2.1) in such a manner that we substitute the function f_1 for function f or in other words, if $ds^2 = 2e^1e^2 + 2e^3e^4$, where the forms e^a are given by the relations (2.1), then

$$ds_1{}^2 \underset{\text{def}}{=} ds^2 - \frac{e\bar{e}}{r^2 + \beta^2}(e^3)^2 \tag{2.16b}$$

Of course the metric ds_1^2 belongs to class (2.1), as the function f is a disposable one. Let

$$G_{ab} \underset{\text{dif}}{=} R_{ab} - \tfrac{1}{2} g_{ab} R \tag{2.17}$$

$$P_{ab} \underset{\text{dif}}{=} 2F_{ac}F_b{}^c - \tfrac{1}{2} g_{ab} F_{cd} F^{cd} \tag{2.18}$$

The following theorem is fulfilled:

Theorem 1. The fulfilment of any Einstein equations

$$G_{ab} = T_{ab} \tag{2.19}$$

by the metric ds^2 belonging to class (2.1) is equivalent to the metric ds_1^2 satisfying the Einstein-Maxwell equations

$$G_{ab} = T_{ab} - P_{ab} \tag{2.20}$$

the electromagnetic field of which is determined by the relations (2.13).

The proof is based on the calculation of the expressions for the tetrad components G_{ab} of the Einstein tensor in both space-times ds^2 and ds_1^2 making use of (1.15) and (2.8). Let us denote these expressions for short by $G_{ab}(ds^2)$ and $G_{ab}(ds_1^2)$. From the calculations mentioned it follows that $G_{ab}(ds_1^2) = G_{ab}(ds^2) - P_{ab}(e, \bar{e})$, where by $P_{ab}(e, \bar{e})$ we have denoted the expressions for the components P_{ab} obtained by substituting in the right-hand sides of relations (2.18) the expressions given in the right-hand sides of equations (2.13) for the corresponding terms F_{cd}. It is easily seen that the thesis in both directions results immediately from the above fact as well as from the fact that the field (2.13) fulfills equations (2.14) in every space-time of class (2.1), to which belongs also the space-time with metric ds_1^2.

Let us note that in the light of (1.10) and (2.8e), $T_{22} = T_{24} = T_{44} = 0$ is the necessary condition for the metric of class (2.1) being among the solutions of equations (2.19). The consistency conditions with respect to the metrics ds_1^2 and equations (2.20) are fulfilled, since from (2.18) considering (2.13d) it follows that $P_{22} = P_{24} = P_{44} = 0$.

Theorem 1 is an equivalence, one of the implications being of [viz., if (2.19) then (2.20)] practical importance, since it says that if we have a solution of any Einstein equations (2.19) for which we can prove that it can be brought to the (2.1) form, then this solution can automatically be extended to include the solution of the Einstein-Maxwell equations (2.20) by adding to the expression for function f the component $-e\bar{e}$, where $e(Y)$ is an arbitrarily chosen analytical function. The class of metrics (2.1) includes generalizations of many known solutions of Einstein equations, among others generalizations of the of the Kerr solution. The generality of class (2.1) seems to be quite high owing to the occurrence in the form e^4 [see (2.1c) and (2.1e)] of a disposable real function γ of all four coordinates.

3. *Comparison with the Works of Other Authors*

Robinson et al. (1969a) have shown that if in any space-time a null vector field k_μ exists that is geodetic and shear-free, then there exists such a system of coordinates $x^\mu = (r, \rho, Y, \bar{Y})$ that the metric form takes the shape

$$ds^2 = 2P^2\, dY\, d\bar{Y} + 2k_\mu dx^\mu (dr + Q\, dY + \bar{Q}\, d\bar{Y} + Sk_\mu dx^\mu) \quad (3.1a)$$

where

$$k_\mu dx^\mu = p(d\rho + q\, dY + \bar{q}\, d\bar{Y}) \quad (3.1b)$$

where the functions p, q may depend only on three variables $\rho, Y, \bar{Y}$, whereas the functions P, Q, S may depend on all four coordinates. As may be seen, the class of metrics (2.1) is a subclass of a more general class (3.1). In the present work we approached the problem in the opposite way. We assumed the form of the metric and therefrom we obtained as a conclusion that vector $e^3{}_\mu$ [corresponding to vector k_μ in (3.1)] is geodetic and shear-free. This situation is a particular case of a more general regularity. It appears that a theorem reverse to the one proved by Robinson et al. (1969a) holds, viz., if we assume that the metric has the form (3.1), then k_μ is the null vector, geodetic, and shear-free. To prove this it suffices to observe that the metric form (3.1). has the shape (1.11) and fulfills the conditions (1.9) and (1.10) if we assign to the vector $e^3{}_\mu$ the vector k_μ assuming at the same time quite formally that $P^2 = P'\bar{P}'$. Calculating in this tetrad system the values of the relevant Γ_{abc} we find that $\Gamma_{424} = \Gamma_{422} = 0$, which proves the thesis. Hence we have the following theorem:

> *Theorem 2.* For every space-time its metric ds^2 can be presented in the form (3.1) if and only if a null, geodetic, shear-free vector field k_μ exists in that space-time.

Robinson et al. (1969a), and Robinson and Robinson (1969) have studied the properties of the (3.1) class of metrics as the solutions of equations $R_{\mu\nu} = 0$. Robinson et al. (1969b) have forwarded a method of expanding solutions of equations $R_{\mu\nu} = 0$ in class (3.1) to the solutions of Einstein-Maxwell equations with electromagnetic field revealing the principal null direction k_μ. In this situation all the results that correspond to the solutions of equations $R_{\mu\nu} = 0$ within the framework of class (2.1)—in the case of theorem 1 when in equations (2.19) and (2.20) we have $T_{ab} = 0$—are particular cases of more general results of the authors quoted. Outside the region of results covered by these authors are those results given in the present work which correspond to the solutions of equations (2.19) and (2.20) with $T_{ab} \neq 0$ for at least one pair of indexes (a, b). Owing to the postulated generality of the γ function such solutions exist in class (2.1), e.g., solutions including the cosmological constant.

The electromagnetic field (2.13) described in Theorem 1 has the potential V_μ given by (2.10), which is a generalization of the potential of the electromagnetic field that was the object of interest of Debney et al. (1969). The potential considered by the latter authors [of appearance analogous to (2.10), cf. Debney et al., 1969, p. 1853 formula (6.8)] in its particular case, when in terms of the present work $e = \bar{e} = \text{const}$, gives a generalization of the Kerr metric to the Kerr-Newman one whose electromagnetic field (1) has a potential that passes into the Coulomb potential of the electric charge e when the Kerr parameter a (according to the notation of Debney et al., 1969) tends to zero, and (2) in the asymptotic area is a Coulomb electric field of charge e and magnetic field of the magnetic dipole with momentum $a \cdot e$ [cf. Debney et al., 1969, pp. 1853–4). The above analogies allow what is from the physical point of view a reasonable interpretation of the $e(Y)$ function in $ds_1{}^2$ metrics, especially those that are generalizations of the Reissner-Nordström metric. In special cases, when $e = e_0 + ig_0 = \text{const}$, the real constants e_0, g_0 can be interpreted as electric and magnetic monopoles, respectively. Let us observe, however, that the separation into electric and magnetic monopoles is significant at the level of the V_μ and F_{ab} quantities characterizing the electromagnetic field [see (2.10) and (2.13)], but the metric $ds_1{}^2$ itself is from the formal point of view insensitive to this separation, since it includes the expression $-e\bar{e}$ equal to $-e_0{}^2 - g_0{}^2$ for $e = e_0 + ig_0$. If therefore $e_0, g_0 = \text{const}$, then the constant $-e_0{}^2 - g_0{}^2$ can be formally replaced in the given metric $ds_1{}^2$ by one nonpositive constant.

4. *Conditions of Solutions in a Certain Subclass of Class* (2.1) *and Explicit Solutions*

In this section we shall concentrate on the solutions of Einstein equations containing the cosmological constant λ:

$$R_{ab} = \lambda g_{ab} \tag{4.1}$$

restricting ourselves to solutions in class (2.1) with the additional limiting assumption

$$\partial_\rho \gamma = \partial_\rho f = 0 \tag{4.2}$$

Of course every such solution may be generalized, in accordance with Theorem 1, to the solution of equations (2.20) with $T_{ab} = -\lambda g_{ab}$, by supplementing the expression for function f with the component $-e(Y)\bar{e}(\bar{Y})$ corresponding to the electromagnetic field (2.13).

After substituting the expressions for R_{ab} from (2.8) to (4.1) and calculating the sum $R_{12} + R_{34}$ we obtain that $\partial_r \partial_r f = 2\epsilon - 4\alpha^2 \partial_Y \partial_{\bar{Y}} \ln \alpha - 4\lambda(r^2 + \beta^2)$. From the latter equation after integration and the integration equations (4.1) [taking account of (4.2)] we get

$$\begin{aligned} f_1 &= r^2(\epsilon - 2\alpha^2 \partial_Y \partial_{\bar{Y}} \ln \alpha) + r(M + \bar{M} - m - \bar{m}) \\ &\quad - i\beta(\bar{m} - m) + \epsilon\beta^2 + 2\alpha^2\beta \partial_Y \partial_{\bar{Y}} \beta \\ &\quad + 2\alpha^2\beta^2 \partial_Y \partial_{\bar{Y}} \ln \alpha + \lambda(-\tfrac{1}{3}r^4 - 2r^2\beta^2 + \beta^4) - e\bar{e} \end{aligned} \tag{4.3}$$

and the relations between the functions α, β, m:

$$\alpha^2 \partial_Y \partial_{\bar{Y}} \beta + 2\alpha^2 \beta \partial_Y \partial_{\bar{Y}} \ln \alpha + \tfrac{4}{3}\lambda\beta^3 = (i/2)(\bar{M} - M) \tag{4.4a}$$

$$\partial_Y \partial_{\bar{Y}}(\alpha^2 \partial_Y \partial_{\bar{Y}} \ln \alpha + \lambda\beta^2) = 0 \tag{4.4b}$$

$$\alpha^2 [(\bar{m} - m) \partial_Y \partial_{\bar{Y}} \ln \alpha + i\epsilon \, \partial_Y \partial_{\bar{Y}} \beta] + i\beta = 0 \tag{4.4c}$$

where $M = M(Y)$ is an arbitrary analytical function of one variable, Y. Equation (4.4c) is the condition (2.1k) added here for completeness. From relations (4.3), (2.16), and (2.1e) we see that in the expression determining the function γ certain components cancel each other. If we provide for the presence of the electromagnetic field (2.13), the expression for γ will take the form

$$\gamma = (r^2 + \beta^2)^{-1} \cdot [-r^2\alpha^2 \partial_Y \partial_{\bar{Y}} \ln \alpha + \alpha^2 \beta \partial_Y \partial_{\bar{Y}} \beta + \alpha^2\beta^2 \partial_Y \partial_{\bar{Y}} \ln \alpha + (r/2)(M + \bar{M}) + (\lambda/2)(-\tfrac{1}{3}r^4 - 2r^2\beta^2 + \beta^4) - \tfrac{1}{2}e\bar{e}] \tag{4.5}$$

Among the limiting cases when $\lambda = 0$, the following solution of equations (4.4) deserves our attention:

$$\alpha = (1/\sqrt{2})(1 + \epsilon Y\bar{Y}) \tag{4.6a}$$

$$\beta = \frac{\epsilon}{2}\left[i(\bar{m} - m) + Y\partial_Y a + \bar{Y}\partial_{\bar{Y}}\bar{a} + \frac{1 - \epsilon Y\bar{Y}}{1 + \epsilon Y\bar{Y}}(a + \bar{a})\right] \tag{4.6b}$$

where $a = a(Y)$ is an arbitrary analytical function of one variable, Y. It can be easily shown that in this case we have

$$m(Y) = M(Y) \tag{4.6c}$$

$$f_1 = -e(Y)\bar{e}(\bar{Y}) \qquad (f = 0) \tag{4.6d}$$

The metric (4.6) without the electromagnetic field has been found earlier by Robinson et al. (1969a) and subsequently expanded by Robinson et al. (1969b) to provide full solution (4.6) with the field (2.13). In the case $\epsilon = 1$, the metric (4.6) is a generalization by means of three arbitrary analytical functions $m(Y)$, $a(Y)$, and $e(Y)$ of the solution found by Demiański and Newman (1966), which in turn is a generalization of the Kerr-Taub-NUT-Newman metric, which consists in supplementing the latter metric with a magnetic charge parameter. The metric mentioned found by Demiański and Newman (1966) is derived from (4.6) by assuming $\epsilon = 1$, $m = m_0 + in_0 = \text{const}$, $a = a_0 = \bar{a}_0 = \text{const}$ [as may be seen from (4.6b), for $a = \text{const}$ only $\operatorname{Re} a$ appears in metric (4.6)], and $e = e_0 + ig_0 = \text{const}$. The constants m_0, n_0, a_0, e_0, g_0 are then the mass, NUT, Kerr, electric charge and magnetic charge parameters, respectively. If in this case a transformation of the coordinates is performed: $Y = \tan(\vartheta/2) \cdot \exp i\varphi$, $\rho = \rho' - 2m_0 \ln \alpha + 2n_0\varphi$, then we obtain the standard axially symmetric form of ds^2 in spatial spheric coordinates r, ϑ, φ with the fourth coordinate ρ'.

Since the solution (4.6) is a generalization of physically interesting metrics, we shall present several properties of metric (4.6) that have not been given by its first discoverers (Robinson et al., 1969a, b).

The metric (4.6) has the following features: It may be only flat or of [2, 2] or [2, 1, 1] Petrov type; it is flat if and only if $C^{(3)} = 0$; $C^{(3)} = 0$ if and only if $m = e = 0$; the metric is of [2, 2] Petrov type if and only if $\partial_Y m = \partial_Y e = 0$ and $m \neq 0$ or $e \neq 0$ and $a = a_1 Y^{-1} + a_2 + a_3 Y$, where a_1, a_2, a_3 are arbitrary complex constants (in view of (4.6b) we can assume without any loss of generality that constant a_2 is real).

The proof of the above properties is based on relations (2.9) and Lemma 1, and in the case of [2, 2] type first it is shown that $\partial_Y m = \partial_Y e = 0$ and subsequently it is proved that the function a has the above-given form. However, the constants a_1, a_2, a_3 do not form a significant system of constants for the solution of [2, 2] type. There exists eventually such a transformation of coordinates, $r, \rho, Y, \bar{Y} \to r, \rho', Y', \bar{Y}'$, where $Y' = (A + BY)/(C + DY)$, $\rho' = \rho - 2m \ln(DY' - \bar{B}) - 2\bar{m} \ln(\bar{D}\bar{Y}' - \bar{B})$, the relation $|AD - BC| = 1$ assumed for the complex constants A, B, C, D so that in the case of [2, 2] type the metric (4.6) preserves its shape of form ds^2, and $a(Y') = \text{const}$. Hence, after performing this sort of transformation and dropping the primes in ρ', Y', $\bar{Y}'$ we have such a situation as if we had not performed the transformation but only assumed that $a(Y) = a_2$ and $a_1 = a_3 = 0$.

In the case of nonvanishing cosmological constant λ two solutions of equations (4.4) were found giving metrics presumably unknown hitherto.

The first one is

$$\alpha = (1/\sqrt{2})(1 + \epsilon Y\bar{Y}) \tag{4.7a}$$

$$\beta = n_0 = \text{const} \tag{4.7b}$$

From (4.4) we then find that $M, m = \text{const}$ and $\operatorname{Im} m = \epsilon n_0$. Putting

$$M_0 \underset{\text{dif}}{=} \operatorname{Re} M = \text{const} \tag{4.7c}$$

we find from (4.3) that

$$f_1 = 2r(M_0 - \operatorname{Re} m) + \lambda(-\tfrac{1}{3}r^4 - 2n_0^2 r^2 + n_0^4) - e(Y)\bar{e}(\bar{Y}) \tag{4.7d}$$

For $\epsilon = 1$ we obtain after transformation of coordinates: $Y = \tan(\vartheta/2) \cdot \exp i\varphi$, $\rho = \rho' - 2 \operatorname{Re} m \cdot \ln \alpha + 2n_0\varphi$, and dropping the prime in ρ' that

$$\begin{aligned} ds^2 &= (r^2 + n_0^2)(d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2) + 2\,dr(d\rho + 2n_0 \cos\vartheta \, d\varphi) \\ &\quad - \left[1 - \frac{2M_0 r + 2n_0^2 + \lambda(-\frac{1}{3}r^4 - 2n_0^2 r^2 + n_0^4) - e(Y)\bar{e}(\bar{Y})}{r^2 + n_0^2}\right] \\ &\quad \times (d\rho + 2n_0 \cos\vartheta \, d\varphi)^2 \end{aligned} \tag{4.8}$$

For $\epsilon = -1$ we obtain after transformation of the coordinates $Y = \tanh(\vartheta/2) \cdot \exp i\varphi$, $\rho = \rho' - 2 \operatorname{Re} m \cdot \ln \alpha - 2n_0\varphi$, and dropping the prime in ρ' that

$$\begin{aligned} ds^2 &= (r^2 + n_0^2)(d\vartheta^2 + \sinh^2 \vartheta \, d\varphi^2) + 2\,dr(d\rho - 2n_0 \cosh\vartheta \, d\varphi) \\ &\quad + \left[1 + \frac{2M_0 r - 2n_0^2 + \lambda(-\frac{1}{3}r^4 - 2n_0^2 r^2 + n_0^4) - e(Y)\bar{e}(\bar{Y})}{r^2 + n_0^2}\right] \\ &\quad \times (d\rho - 2n_0 \cosh\vartheta \, d\varphi)^2 \end{aligned} \tag{4.9}$$

The solution (4.8) is the generalization, given in standard form in spheric coordinates r, ϑ, φ, with ρ as fourth coordinate, of the Taub-NUT metric providing for the presence of the cosmological constant λ[4] and electromagnetic field (2.13). The constants M_0, n_0 are parameters of mass and NUT, respectively, whereas $e(Y)$ is an arbitrary analytical function determining the electromagnetic field. The metric (4.9) is a hyperbolic equivalent of the metric given by (4.8).

Metric (4.7) reveals the following properties: It is conformally flat if and only if $C^{(3)} = 0$; $C^{(3)} = 0$ if and only if $M_0 = n_0 = e = 0$; it is of [2,2] Petrov type if and only if $C^{(3)} \neq 0$ and $\partial_Y e = 0$; is of [2, 1, 1] Petrov type if and only if $\partial_Y e \neq 0$.

The proof is based on the relations (2.9) and Lemma 1.

Besides, calculating $C^{(i)}$ from (2.9) and Γ_{31a} from (2.6), it may be easily shown that for metric (4.7) the condition $\partial_Y e = 0$ is equivalent to the fact that vector $e^{4\mu}$ is geodetic, shear-free, and reveals an expansion different from zero; the rotation of vector $e^{4\mu}$ vanishes if and only if $n_0 = 0$; the condition that $C^{(3)} \neq 0$ and $\partial_Y e = 0$ ([2, 2] type) is equivalent to the vector $e^{4\mu}$ being a double Debever-Penrose vector. Hence, in the case when metric (4.7) is not of [2, 1, 1] type, the independent vectors $e^{3\mu}$ and $e^{4\mu}$ have the same features from among those mentioned hitherto.

The second solution of equations (4.4) with nonvanishing cosmological constant which has been found has the form

$$\lambda > 0 \tag{4.10a}$$

$$\alpha = (2\lambda)^{1/2}\left(\int H\,dY + \int \bar{H}\,d\bar{Y}\right) \tag{4.10b}$$

$$\beta = i(\bar{H} - H) \tag{4.10c}$$

where $H = H(Y)$ is an arbitrary analytical function of one variable, Y. From (4.4) we get the following relations between the functions H, m, and M:

$$H = [2\lambda(m - c_1)]^{-1} = [(3/8\lambda)(c_0 - M)]^{1/3} \tag{4.10d}$$

where c_0, c_1 are arbitrary real constants. Considering (4.10d) we can select one of the functions H, m, M as an arbitrary one. It is most convenient to choose H. In that case we obtain from relation (4.5), already providing for the presence of the electromagnetic field, that

$$\begin{aligned}\gamma = [r^2 - (H - \bar{H})^2]^{-1} \cdot \{&c_0 r - \tfrac{1}{2}e\bar{e} + \lambda[-\tfrac{1}{6}r^4 + r^2(H^2 + \bar{H}^2)\\ &- \tfrac{4}{3}r(H^3 + \bar{H}^3) + \tfrac{1}{2}(H^2 - \bar{H}^2)^2]\}\end{aligned} \tag{4.10e}$$

After transformation of the coordinates $r = r' + H + \bar{H}$, $\rho = \rho' - 2c_1 \ln\alpha + \epsilon(H + \bar{H})$, and subsequently dropping the primes at r' and ρ' we obtain the

[4] The generalization of the Taub-NUT metric (with signature $+ - - -$) to the solution with a cosmological constant (without the electromagnetic field) has been given earlier by Demiański (1972 and 1973), but the term providing for the presence of the cosmological constant λ has been calculated by him incorrectly. Eventually in a system of coordinates identical as in (4.8) (account being taken of the signature) Demiański has given $-(\lambda/3)(r^2 + 5n_0^2)\cdot(e^3)^2$, whereas this term should be $-\lambda(\frac{1}{3}r^4 + 2n_0^2r^2 - n_0^4)(r^2 + n_0^2)^{-1}\cdot(e^3)^2$.

metric $ds^2 = 2e^1e^2 + 2e^3e^4$, where

$$e^1 = \frac{r + 2H}{(2\lambda)^{1/2}} [\int H dY + \int \bar{H} d\bar{Y}]^{-1} dY = \overline{e^2} \tag{4.11a}$$

$$e^3 = d\rho + \lambda^{-1} [\int H dY + \int \bar{H} d\bar{Y}]^{-1} d(Y + \bar{Y}) \tag{4.11b}$$

$$e^4 = dr + [r^2 + 2r(H + \bar{H}) + 4H\bar{H}]^{-1} \times \{c_0(r + H + \bar{H}) - (\lambda/6)r^2[r^2 + 4r(H + \bar{H}) + 12H\bar{H}] - \tfrac{1}{2}e\bar{e}\}e^3 \tag{4.11c}$$

The metric presented above corresponds to both values of the coefficient $\epsilon = \pm 1$, and, as is seen, it is independent of ϵ. In metric (4.11) the limiting transition $\lambda \to 0$ cannot be realized if $H \neq 0$. However, if we assume formally that $\lambda^{1/2} \int H dY = h = \text{const}$ [Re $h \neq 0$ considering (2.1g) and (4.10b)], then $H = 0$, and after transformation of the coordinates $\rho = \rho' - (Y' + \bar{Y}')\lambda^{-1/2}$, $Y = (h + \bar{h})Y'$, and dropping the primes at ρ', Y' we obtain a subcase of metric (4.11):

$$ds^2 = r^2 dY d\bar{Y} + 2dr\, d\rho + (1/r^2)[2c_0 r - (\lambda/3)r^4 - e\bar{e}]\, d\rho^2 \tag{4.12}$$

in which the limiting transition $\lambda \to 0$ is of course admissible.

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